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Numerical modeling of Darcy–Weisbach friction factor and branching pipes problem

Tefaruk Haktanır*, Mehmet Ardıçlıoğlu

Department of Civil Engineering, Erciyes University, 38039 Kayseri, Turkey

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Abstract

First, a numerical algorithm for the friction factor in the Darcy–Weisbach pipe friction head loss formula is developed by an effective linear iteration scheme of the Colebrook–White equation, which precisely determines, with a small load of computations, the friction factor within the ranges of: $0 <$ relative roughness < 0.1 and $2\times10^3 <$ Reynolds number $< 10^9$. The developed subroutine can be adapted to any pipe friction loss parts of any pipe network problems. Next, the branching pipes problem is formulated as a system of non-linear equations, and an efficient, practical, and always convergent numerical algortihm for its solution is developed, in which the Darcy–Weisbach equation is used for the friction losses, the friction factor being computed by the above-mentioned algorithm. The model can handle many reservoirs which are interconnected by pipes branching from a common junction in just a couple of seconds of execution time.

Keywords: Darcy–Weisbach friction factor; Branching pipes problem

1. Introduction

The Darcy–Weisbach equation for the friction loss in pipes yields better precision than other equations such as Hazen–Williams because the friction factor it involves is determined as a function of both the relative pipe wall roughness ($e=k/D$) and the Reynolds number (Re). The others, like Hazen–Williams, Manning, and Scobey assume that the flow is in the rough pipe zone and neglect the effect of Re. Comments like: “Each of them is applicable only to problems involving flow of water at normal temperatures and at a relatively high degree of turbulence, as well as to ordinary commercial pipes.”, which appears on page 73, and like: “They are based on data obtained at fairly high Reynolds numbers, with therefore a high degree of turbulence.” appearing on page 77 of the classical book by Morris and Wiggert [4], can be found in many relevant sources. Formerly, the friction factor (λ) in the Darcy–Weisbach equation was obtained graphically from the Moody diagram, which contains so many lines for so many combinations of relative roughness (e) and Re. This manual procedure is time-consuming and may not be precise. The Colebrook–White formula [1] can be used as a replacement for the entire Moody diagram, as it very closely simulates all the curves of the Moody diagram almost exactly. However, it is a formula in open form which does not lend itself for easy computation of λ. Hence, one of the objectives herein has been to devise a practical algorithm for the correct λ as a function of both e and Re.

The branching pipes problem comprises three or more number of reservoirs interconnected by pipes branching from a common junction point for which a trial-and-error solution is advocated in all the conventional hydraulics textbooks [1,3,5,6]. Streeter et al. advocate using an optimizer on a spreadsheet in order to perform these trial-and-error computations more conveniently [5]. Hence, formulation of the branching pipes problem in a more sophisticated manner eliminating any trial-and-error computations with no user interference has been another objective of this study.

2. Determination of the Darcy–Weisbach friction factor

The Darcy–Weisbach equation is considered to calculate the head loss due to friction through a pipe more realistically

* Corresponding author.
E-mail addresses: thaktan@erciyes.edu.tr (T. Haktanır), mardic@erciyes.edu.tr (M. Ardıçlıoğlu).

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Nomenclature

c_i	discharge coefficient, which is +1 for the discharging and −1 for the filling reservoirs
Ce_i	coefficient of energy loss of exit from the *i*th reservoir
ΣC_{li}	summation of minor loss coefficients over the *i*th pipe such as bends and valves
D_i	diameter of the *i*th pipe (m)
ΔH	increment of value for the hydraulic head at the junction computed at the end of one cycle of Newton–Raphson iterative algorithm (m)
ΔV_i	increment of velocity in the *i*th pipe computed at the end of one cycle of Newton–Raphson iterative algorithm (m/s)
e	relative roughness of any pipe
λ	Moody pipe friction factor
g	acceleration of gravity (m/s^2)
H	hydraulic head at the junction (m), and
J	junction point at which all the branching pipes meet
k	height of nominal roughness of interior of pipe wall (m)
L_i	length of the *i*th pipe (m)
Q_i	low rate through the *i*th pipe (m^3/s)
Re	Reynold's number
s_{n+1}	symbol used for the left hand side of Eq. (10) computed with the latest values of V_i's and inserted in Eq. (15) as the last element of the load vector
t_i	symbol used for the left hand side of Eq. (3) for the *i*th pipe
t'_{ii}	first partial derivative of t_i with respect to the *i*th variable
u_i	symbol used for the left hand side of Eq. (4) for the *i*th pipe
u'_{jj}	first partial derivative of u_i with respect to the *i*th variable
V_{i_0}	velocity in the *i*th pipe computed for the hypothetical case of hydraulic head at the junction equalling water surface elevation of the *j*th reservoir (m/s)
V_{i_1}	initial estimate for the velocity in the *i*th pipe
V_{i_2}	next value for the velocity in the *i*th pipe computed at the end of one cycle of Newton–Raphson iterative algorithm (m/s)
Z_i	water surface elevation of the *i*th reservoir (m)
ν	kinematic viscosity of water flowing in the pipe (m^2/s)

because it takes into account the smooth pipe, transition flow, and rough pipe flow cases. Determination of its pipe friction factor, λ, as a function of both the relative roughness, e and Reynolds number, Re, out of the well-known Moody diagram, which has been developed as the outcome of long and tedious accumulation of prototype experiments, provides this precision. As an alternative to so many curves of the Moody diagram, the Colebrook–White formula has been proposed [2], which is presented in many relevant publications like Brater and King [1], also, as:

$$\lambda^{-0.5} + 2\log_{10}\{e/3.7 + (2.51/Re)\lambda^{-0.5}\} = 0 \quad (1)$$

where,

- λ Darcy–Weisbach pipe friction factor,
- e relative roughness = nominal roughness height (mm)/inner pipe diameter (mm), k/D,
- Re Reynold's number of the flow conveyed in the pipe = VD/ν, and
- ν kinematic viscosity of water flowing in the pipe (m^2/s)

Computation of λ for given e and Re with the help of Colebrook–White equation requires an iterative method. In this study, Newton–Raphson, secant, and linear iteration methods of numerical root-finding have been applied on that equation, and it has been realized that in this case the linear iteration method is suprior to the other two, although they are more commonly known and more popular in general. Both the Newton–Raphson and secant algorithms for this problem necessitate an initial estimate for λ which must be smaller than the corrrect value. Otherwise, for some combinations of e and Re, after a few steps the iterations stop with the comment: "Attempt of taking square-root of negative argument".

Colebrook–White formula can be manipulated as:

$$\lambda = \{-1.15/\ln[e/3.7 + (2.51/Re)\lambda^{-0.5}]\}^2 \quad (2)$$

When the iterations are heading towards the root of an equation by any recursive method, the sequential differences between the consecutive pairs of iterations must approach zero. In other words, for convergence, it is obvious that $|x_i - x_{i-1}|$ must be smaller than $|(x_{i-1} - x_{i-2})|$. It is a known fact that there is a straightforward analytical relationship between the last three iterations by the linear iteration method, which is:

$$x_i - x_{i-1} = (x_{i-1} - x_{i-2})|g'(x)| \quad (3)$$

By this equation, in order for $|x_i - x_{i-1}|$ to be smaller than $|(x_{i-1} - x_{i-2})|$, the term: $|g'(x)|$ must be smaller than one. It is also obvious by Eq. (3) that closer the magnitude of the term: $|g'(x)|$ to zero, greater the difference between $|x_i - x_{i-1}|$ and $|(x_{i-1} - x_{i-2})|$, which means faster the rate of convergence.

In our case, $g'(x)$ is:

$$g'(x) = [(1.3255)(2.51/Re)\lambda^{-1.5}]/\{\{\ln[e/3.7 + (2.51/Re)\lambda^{-0.5}]\}^3[e/3.7 + (2.51/Re)\lambda^{-0.5}]\} \quad (4)$$

which assumes values always very close to zero for any λ in the possible range of: $0.002 < \lambda < 0.1$, with any combinations of e and Re values. The numerical values of $|g'(x)|$ are always smaller than 0.1 and mostly even smaller than 0.01, which can be verified using Eq. (4) by anyone interested. Therefore the rate of convergence of the linear iteration method by Eq. (2) is very fast, and it is almost the same as and even a little faster than those of Newton–Raphson and secant methods. The total amount of the arithmetic operations for each iteration of the linear iteration method is also less than that by either the Newton–Raphson or the secant method. Generally, λ is within $0.01 < \lambda < 0.03$, and therefore, starting out the iterations with the initial value of: $\lambda_1 = 0.02$, and inserting this in the right hand side of Eq. (2), the iterations of: $\lambda_2 = g(\lambda_1)$ and new λ_1 = previous λ_2 yields the correct λ in just a few cycles even for extreme λ values.

3. Branching pipes problem

Fig. 1 depicts the problem of branching pipes interconnecting n number of reservoirs, which are represented by integer numbers, the 1st reservoir being the highest and the nth one the lowest, and the ith pipe connecting the junction, J, to the ith reservoir. Flow in the 1st pipe is from the 1st reservoir towards J because the water surface elevation of the uppermost reservoir is the highest. Similarly, the direction of flow in the nth pipe is always towards the nth, the lowermost reservoir. Flow in any intermediate pipe may be either to or from the reservoir to which it is connected. Velocities and directions of flows in all the pipes need to be determined. The hydraulic head at the junction is another unknown. Application of the Bernoulli equation between the water surface elevations of the reservoirs and the junction produces n number of energy equations. The continuity equation at the junction constitutes the $n+1$ independent equation.

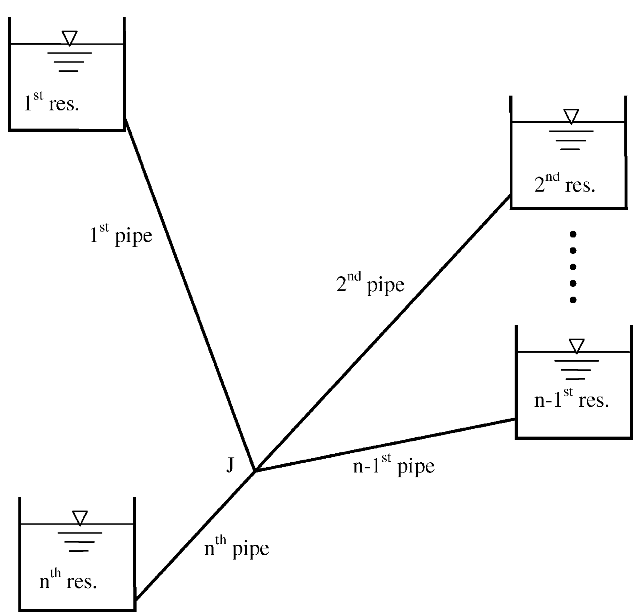

Fig. 1. Branching pipes probem.

The energy equations between the water surface elevation of a discharging reservoir and the junction, and between the junction and the water surface elevation of a filling reservoir over the connecting pipes are:

$$Z_i - C_{ei}\frac{V_i^2}{2g} - \sum C_{li}\frac{V_i^2}{2g} - \lambda_i\frac{L_i}{D_i}\frac{V_i^2}{2g} = H \quad (5)$$

and

$$H - \sum C_{li}\frac{V_i^2}{2g} - \lambda_i\frac{L_i}{D_i}\frac{V_i^2}{2g} - \frac{V_i^2}{2g} = Z_i \quad (6)$$

respectively. In these equations,

- Z_i water surface elevation of the ith reservoir (m),
- C_{ei} coefficient of energy loss of exit from the ith reservoir,
- ΣC_{li} summation of minor loss coefficients over the ith pipe such as bends and valves,
- λ_i pipe friction factor of the ith pipe in the Darcy–Weisbach friction loss formula,
- L_i length of the ith pipe (m),
- D_i diameter of the ith pipe (m),
- V_i average flow velocity in the ith pipe (m/s),
- H hydraulic head at the junction (m), and
- g acceleration of gravity (m/s^2).

Eqs. (5) and (6) can be rewritten as:

$$\frac{V_i^2}{2g}\left(C_{ei} + \sum C_{li} + \lambda_i\frac{L_i}{D_i}\right) + H - Z_i = 0 \quad (7)$$

and

$$\frac{V_i^2}{2g}\left(\sum C_{li} + \lambda_i\frac{L_i}{D_i} + 1\right) - H + Z_i = 0 \quad (8)$$

For the 1st reservoir Eq. (7) is valid, and for the nth reservoir Eq. (8) applies. Either equation is valid for the intermediate reservoirs.

The continuity equation at the junction is:

$$Q_1 + \sum_{i=2}^{n-1} c_i Q_i - Q_n = 0 \quad (9)$$

where,

- Q_i flow rate through the ith pipe (m^3/s), and
- c_i discharge coefficient, which is +1 for the discharging reservoirs, and −1 for the filling reservoirs.

If all the pipes are circular, Eq. (9) can be rewritten as:

$$V_1 D_1^2 + \sum_{i=2}^{n-1} c_i V_i D_i^2 - V_n D_n^2 = 0 \tag{10}$$

Those reservoirs whose water surface levels are higher than hydraulic head of the junction, H, are discharging, and those whose water surface levels are lower than H are filling. Symbolically, if $j-1$ is the number of the lowest discharging reservoir, as the others from the jth down to the nth are filling, then the below inequality holds:

$$Z_j < H < Z_{j-1} \tag{11}$$

Denoting the left hand sides of Eqs. (7) and (8) by t and u, respectively, the branching pipes problem for which inequality 11 holds can be depicted as a system of $n+1$ number of nonlinear equations as:

$$\begin{aligned} &t_1 = 0 \\ &\vdots \\ &t_{j-1} = 0 \\ &u_j = 0 \\ &u_{j+1} = 0 \\ &\vdots \\ &u_n = 0 \\ &D_1^2 V_1 + \cdots + D_{j-1}^2 V_{j-1} - D_j^2 V_j - \cdots - D_n^2 V_n = 0 \end{aligned} \tag{12}$$

where, the velocities and hydraulic head of the junction become $n+1$ number of unknowns.

3.1. Determination of discharging and filling reservoirs

A loop of computations is executed to determine the discharging and filling reservoirs. Firstly j is assumed to equal 2, which implies that the rest of the $n-2$ number of lower reservoirs except for the uppermost one are filling. Then, the velocity in the pipe of a discharging reservoir is computed directly by the following equation:

$$V_i = \left[2g(Z_i - H)/\left(C_{ei} + \sum C_{li} + \lambda_i L_i/D_i\right)\right]^{0.5} \tag{13}$$

And, the velocity in the pipe of a filling reservoir is computed directly by the following equation:

$$V_i = \left[2g(H - Z_i)/\left(\sum C_{li} + \lambda_i L_i/D_i + 1\right)\right]^{0.5} \tag{14}$$

In Eqs. (11) and (12), H is set equal to the water surface elevation of the jth reservoir. The discharges in the pipes are computed by multiplying the velocity with the cross-sectional area. If the left hand side of Eq. (9) is positive, then the jth reservoir and the lower ones are filling reservoirs while the $j-1$ and higher ones are the discharging reservoirs. Conversely, if the left hand side of Eq. (9) becomes negative, the discharging flow rates are smaller than the filling flow rates and hence the actual H must be lower than the water surface elevation of the jth reservoir. By lowering H, the flow rates in the discharging pipes will increase while some filling pipes will become discharging pipes and the flow rates of the lower filling pipes will decrease so that the sum of discharging flow rates approach the sum of filling flow rates. Therefore, j is increased by one, meaning the head H, is assumed to equal the water surface elevation of the next lower intermediate reservoir. The velocities in the pipes are computed again with the help of Eqs. (13) and (14) with the new H, and the sign of Eq. (9) is evaluated anew. This loop continues until the value of Eq. (9) turns negative.

3.2. Solution of the system of nonlinear equations

Once the position of the actual H is determined, solution of the system of Eq. (12) is performed by the Newton–Raphson method, which is summarized in the following.

Written in matrix form, the system of linear equations to be solved for the increments of ΔV_i and ΔH is:

$$\begin{bmatrix} t'_{11} & 0 & 0 \ldots & \ldots & 0 & 0 & \ldots 0 & 0 & +1 \\ 0 & t'_{22} & 0 \ldots & \ldots & 0 & 0 & \ldots 0 & 0 & +1 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & t'_{j-1,j-1} & 0 & 0 & \ldots 0 & 0 & +1 \\ 0 & 0 & 0 \ldots & 0 & u'_{jj} & 0 & \ldots 0 & 0 & -1 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots & \ldots & \ldots & \ldots & \ldots 0 & u'_{nn} & -1 \\ D_1^2 & D_2^2 \ldots\ldots & & D_{j-1}^2 & -D_j^2 & -D_{j+1}^2 \ldots\ldots & & -D_n^2 & 0 \end{bmatrix} \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \\ \vdots \\ \Delta V_{j-1} \\ \Delta V_j \\ \vdots \\ \Delta V_n \\ \Delta H \end{bmatrix} = \begin{bmatrix} -t_1 \\ -t_2 \\ \vdots \\ -t_{j-1} \\ -u_j \\ \vdots \\ -u_n \\ -s \end{bmatrix} \tag{15}$$

where,

- s_{n+1} magnitude of the left hand side of Eq. (10) computed with the latest values of the V_i's,
- t'_{ii} partial derivatives of t_i with respect to V_i, and
- u'_{jj} partial derivatives of u_j with respect to V_j.

Next, the improved values for the unknowns are computed by:

$$V_{i_2} = V_{i_1} + \Delta V_i \tag{16a}$$

$$H_2 = H_1 + \Delta i \tag{16b}$$

Denoting the velocities computed by Eqs. (13) and (14) with the correct j as V_{i_0}, the actual velocities in the discharging pipes will be a little smaller than V_{i_0}'s while those in the filling pipes will be a little greater than V_{i_0}'s. Therefore, the initial velocity estimates in the discharging pipes are taken arbitrarily as:

$$V_{i_1} = 0.8V_{i_0} \tag{17}$$

And, the initial velocity estimates in the filling pipes are taken as:

$$V_{i_1} = 1.2V_{i_0} \tag{18}$$

The initial estimate for the velocity in the pipe which is connected to the jth reservoir is arbitrarily taken as 1.0 m/s. Finally, the initial estimate for the $n+1$ unknown, H, is computed by:

$$H_1 = Z_j + 0.2(Z_{j-1} - Z_j) \tag{19}$$

The partial derivatives of any t_i or u_i with respect to $V_1, V_2, \ldots, V_n$, H are zero except for the ith independent variable. The ith partial derivatives of t_i and u_i with respect to V_i are given below:

$$t'_{ii} = V_i\left(C_{ei} + \sum C_{li} + \lambda_i L_i/D_i\right)/g \tag{20}$$

$$u'_{ii} = V_i\left(\sum C_{li} + \lambda_i L_i/D_i + 1\right)/g \tag{21}$$

After having assigned the initial estimates to the n velocities and to H, the system of $n+1$ number of linear equations given as Eq. (15) is solved for the increments, ΔV_i's and ΔH. Next, $n+1$ number of relative differences are computed as shown below:

$$RD_i = |\Delta V_i / V_{i_2}| \quad \text{for } i = 1, \ldots, n$$

$$RD_{n+1} = |\Delta H / H_2|$$

If all these relative differences are less than or equal to 10^{-6}, then the increments are small enough, meaning the convergence is achieved and the solution of the unknown vector is obtained to six significant digit precision. If any of the relative differences are not small enough, then the assignments: $V_{i_1} = V_{i_2}$, $H_1 = H_2$ are made, and the system of linear equations defined as Eq. (15) are computed with these new values. The iterations continue until $V_{i_2} \approx V_{i_1}$ and $H_2 \approx H_1$ within six digit precision.

4. Example problems

The problem given on pages: 556–557 of the book by Streeter et al. [5] is solved using the method described above. The input data and the solution are given in Appendix I. Another example consisting of nine reservoirs is given in Appendix II, in which the roughness heights, lengths, and diameters of the branching pipes are purposely chosen to be extreme values. Still, the developed rouitne converges in six loops.

5. Conclusions

The Colebrook–White equation is solved for the Darcy–Weisbach pipe friction factor by an efficient linear iteration algorithm, which can be adopted for friction loss part of any problem involving pipes. Separately, a numerical model is developed for the branching pipes problem, which eliminates a trial-and-error approach, capable of handling even 100 branching pipes in just a few seconds, in which the pipe friction losses are computed by Darcy–Weisbach equation whose pipe friction factor is computed by the algorithm developed in the first part of the study.

The energy equations considered for the branching pipes problem, Eqs. (7) and (8), take into account all the minor losses such as exit from a discharging reservoir, pipe bends, valves, and entrance to a filling reservoir also along with the pipe friction losses. Modelling of a branching pipe configuration consisting of any number of interconnected reservoirs by the system of the non-linear equations as summarized by Eq. (12), and solution of this system with a high numerical precision automatically in the systematic way depicted, with correct quantification of even the minor losses along with the friction losses, are the improvements to the classical trial-and-error approach advocated in all the relevant sources so far.

Appendix A

Example problem about three branching pipes given on page: 556 in the book by Streeter et al. [5]

Water surface elevations of the reservoirs:

Water surface elevation of reservoir no. 1 = 30.00 m
Water surface elevation of reservoir no. 2 = 18.00 m
Water surface elevation of reservoir no. 3 = 9.00 m

Length, diameter, and roughness height of the pipes:

$L(1) = 3000$ m, $D(1) = 100$ cm, $k(1) = 0.20$ mm
$L(2) = 600$ m, $D(2) = 45$ cm, $k(2) = 0.90$ mm
$L(3) = 1000$ m, $D(3) = 60$ cm, $k(3) = 0.60$ mm

Entrance loss coefficients of exit from reservoirs into the pipes:

$C_e(1) = 0.00$, $C_e(2) = 0.00$

Total minor loss coefficients in the pipes are:

$C_l(1) = 0.00$, $C_l(2) = 0.00$, $C_l(3) = 0.00$, Kinematic viscosity of water, $\nu = 1.00 \times 10^{-06}$ (m^2/s)

V's and Q's in the upper pipes assuming $H = Z(2)$

$V_0(1) = 2.31$ m/s, $Q_0(1) = 1.816$ m^3/s
$V_0(2) = 0.00$ m/s, $Q_0(2) = 0.000$ m^3/s

V's and Q's in the lower pipes assuming $H = Z(2)$

$V_0(3) = 2.27$ m/s, $Q_0(3) = 0.642$ m^3/s

Summation of Q's in the upper pipes = 1.82
Summation of Q's in the lower pipes = 0.64
H actual is between water surface elevations of reservoirs: 1 and 2

$18.00 < H < 30.00$

Iterations for the system of four nonlinear equations:

V_{i_1}	V_{i_2}	λ_i
1.84952	1.63026	0.01418
1.00000	2.74922	0.02000
2.72515	2.98206	0.01986
H_{J_1}	H_{J_2}	
20.40000	24.34336	
⋮		
1.50924	1.50924	0.01439
2.05404	2.05404	0.02362
3.03694	3.03694	0.01981
24.98791	24.98791	

The solution is reached in six loops yielding the following result:
Hydraulic head at the junction point: $H = 24.99$ m
Flow rates incoming to the junction:

$Q(1) = 1.1854$ m^3/s, $V(1) = 1.509$ m/s

Flow rates outgoing from the junction:

$Q(2) = 0.3267$ m^3/s, $V(2) = 2.054$ m/s
$Q(3) = 0.8587$ m^3/s, $V(3) = 3.037$ m/s

Appendix B

Example problem about nine branching pipes for the purpose of demonstration

Water surface elevations of the reservoirs:

WSE of reservoir no. 1 = 109.00 m
WSE of reservoir no. 2 = 99.00 m
WSE of reservoir no. 3 = 88.00 m
WSE of reservoir no. 4 = 77.00 m
WSE of reservoir no. 5 = 66.00 m
WSE of reservoir no. 6 = 55.00 m
WSE of reservoir no. 7 = 44.00 m
WSE of reservoir no. 8 = 33.00 m
WSE of reservoir no. 9 = 22.00 m

Length, diameter, and roughness height of the pipes:

$L(1) = 1570$ m, $D(1) = 40$ cm, $k(1) = 5.80$ mm
$L(2) = 1050$ m, $D(2) = 30$ cm, $k(2) = 0.00$ mm
$L(3) = 800$ m, $D(3) = 25$ cm, $k(3) = 0.15$ mm
$L(4) = 1400$ m, $D(4) = 45$ cm, $k(4) = 0.00$ mm
$L(5) = 700$ m, $D(5) = 40$ cm, $k(5) = 0.20$ mm
$L(6) = 450$ m, $D(6) = 30$ cm, $k(6) = 0.10$ mm
$L(7) = 2500$ m, $D(7) = 25$ cm, $k(7) = 0.75$ mm
$L(8) = 600$ m, $D(8) = 45$ cm, $k(8) = 2.50$ mm
$L(9) = 1700$ m, $D(9) = 40$ cm, $k(9) = 0.00$ mm

Loss coefficients of exit from reservoirs into the pipes:

$C_e(1) = 0.50$, $C_e(2) = 0.50$, $C_e(3) = 0.50$, $C_e(4) = 0.50$, $C_e(5) = 0.60$, $C_e(6) = 0.70$,
$C_e(7) = 0.80$, $C_e(8) = 0.30$

Total minor loss coefficients in the pipes are:

$C_l(1) = 1.50$, $C_l(2) = 2.50$, $C_l(3) = 3.20$, $C_l(4) = 2.30$, $C_l(5) = 0.00$, $C_l(6) = 1.20$,
$C_l(7) = 1.50$, $C_l(8) = 1.60$, $C_l(9) = 2.30$

Kinematic viscosity of water, $\nu = 10^{-6}$ (m^2/s)

V's and Q's in the upper pipes assuming $H_J = Z(2)$

$V_0(1) = 1.07$ m/s, $Q_0(1) = 0.134$ m^3/s
$V_0(2) = 0.00$ m/s, $Q_0(2) = 0.000$ m^3/s

V's and Q's in the lower pipes assuming $H_J = Z(2)$

$V_0(3) = 1.88$ m/s, $Q_0(3) = 0.092$ m^3/s
$V_0(4) = 3.30$ m/s, $Q_0(4) = 0.526$ m^3/s
$V_0(5) = 4.56$ m/s, $Q_0(5) = 0.573$ m^3/s
$V_0(6) = 5.75$ m/s, $Q_0(6) = 0.406$ m^3/s
$V_0(7) = 2.02$ m/s, $Q_0(7) = 0.099$ m^3/s
$V_0(8) = 5.39$ m/s, $Q_0(8) = 0.858$ m^3/s
$V_0(9) = 5.35$ m/s, $Q_0(9) = 0.672$ m^3/s

Summation of Q's in the upper pipes = 0.13
Summation of Q's in the lower pipes = 3.23

V's and Q's in the upper pipes assuming $H_J = Z(3)$

$V_0(1) = 1.55$ m/s, $Q_0(1) = 0.195$ m^3/s
$V_0(2) = 2.22$ m/s, $Q_0(2) = 0.157$ m^3/s
$V_0(3) = 0.00$ m/s, $Q_0(3) = 0.000$ m^3/s

V's and Q's in the lower pipes assuming $H_J = Z(3)$

$V_0(4) = 2.34$ m/s, $Q_0(4) = 0.372$ m^3/s
$V_0(5) = 3.72$ m/s, $Q_0(5) = 0.468$ m^3/s
$V_0(6) = 4.98$ m/s, $Q_0(6) = 0.352$ m^3/s
$V_0(7) = 1.80$ m/s, $Q_0(7) = 0.089$ m^3/s
$V_0(8) = 4.92$ m/s, $Q_0(8) = 0.783$ m^3/s
$V_0(9) = 4.95$ m/s, $Q_0(9) = 0.622$ m^3/s

Summation of Q's in the upper pipes = 0.35
Summation of Q's in the lower pipes = 2.69

V's and Q's in the upper pipes assuming $H_J = Z(4)$

$V_0(1) = 1.91$ m/s, $Q_0(1) = 0.240$ m^3/s
$V_0(2) = 3.14$ m/s, $Q_0(2) = 0.222$ m^3/s
$V_0(3) = 1.88$ m/s, $Q_0(3) = 0.092$ m^3/s
$V_0(4) = 0.00$ m/s, $Q_0(4) = 0.000$ m^3/s

V's and Q's in the lower pipes assuming $H_J = Z(4)$

$V_0(5) = 2.63$ m/s, $Q_0(5) = 0.331$ m^3/s
$V_0(6) = 4.06$ m/s, $Q_0(6) = 0.287$ m^3/s
$V_0(7) = 1.56$ m/s, $Q_0(7) = 0.077$ m^3/s
$V_0(8) = 4.40$ m/s, $Q_0(8) = 0.700$ m^3/s
$V_0(9) = 4.52$ m/s, $Q_0(9) = 0.568$ m^3/s

Summation of Q's in the upper pipes = 0.55
Summation of Q's in the lower pipes = 1.96

V's and Q's in the upper pipes assuming $H_J = Z(5)$

$V_0(1) = 2.22$ m/s, $Q_0(1) = 0.279$ m^3/s
$V_0(2) = 3.85$ m/s, $Q_0(2) = 0.272$ m^3/s
$V_0(3) = 2.66$ m/s, $Q_0(3) = 0.131$ m^3/s
$V_0(4) = 2.35$ m/s, $Q_0(4) = 0.374$ m^3/s
$V_0(5) = 0.00$ m/s, $Q_0(5) = 0.000$ m^3/s

V's and Q's in the lower pipes assuming $H_J = Z(5)$

$V_0(6) = 2.87$ m/s, $Q_0(6) = 0.203$ m^3/s
$V_0(7) = 1.27$ m/s, $Q_0(7) = 0.063$ m^3/s
$V_0(8) = 3.81$ m/s, $Q_0(8) = 0.606$ m^3/s
$V_0(9) = 4.04$ m/s, $Q_0(9) = 0.508$ m^3/s

Summation of Q's in the upper pipes = 1.06
Summation of Q's in the lower pipes = 1.38

V's and Q's in the upper pipes assuming $H_J = Z(6)$

$V_0(1) = 2.48$ m/s, $Q_0(1) = 0.312$ m^3/s
$V_0(2) = 4.44$ m/s, $Q_0(2) = 0.314$ m^3/s
$V_0(3) = 3.26$ m/s, $Q_0(3) = 0.160$ m^3/s
$V_0(4) = 3.33$ m/s, $Q_0(4) = 0.529$ m^3/s
$V_0(5) = 2.65$ m/s, $Q_0(5) = 0.333$ m^3/s
$V_0(6) = 0.00$ m/s, $Q_0(6) = 0.000$ m^3/s

V's and Q's in the lower pipes assuming $H_J = Z(6)$

$V_0(7) = 0.90$ m/s, $Q_0(7) = 0.044$ m^3/s
$V_0(8) = 3.11$ m/s, $Q_0(8) = 0.495$ m^3/s
$V_0(9) = 3.50$ m/s, $Q_0(9) = 0.440$ m^3/s

Summation of Q's in the upper pipes = 1.65
Summation of Q's in the lower pipes = 0.98

H_J actual is between WSE's of reservoirs: 5 and 6
$55.00 < H_J < 66.00$

Iterations for the system of 10 nonlinear equations:

V_{i_1}	V_{i_2}	λ_i
1.98795	2.31646	0.04320
3.55336	4.15196	0.01109
2.61022	2.84121	0.01794
2.66046	2.73712	0.01088
2.12072	1.51026	0.01718
1.00000	2.94266	0.02000
1.08183	1.17920	0.02677
3.73591	3.63927	0.03142
4.20249	4.01438	0.01100
H_{J_1}	H_{J_2}	
57.20000	63.01770	
⋮		
2.29396	2.29396	0.04321
4.08406	4.08406	0.01125
2.82837	2.82837	0.01802
2.69586	2.69586	0.01127
1.37508	1.37508	0.01758
2.43184	2.43184	0.01618
1.17691	1.17691	0.02663
3.63601	3.63601	0.03142
4.05365	4.05365	0.01074
62.97673	62.97672	

The solution is reached in six loops:

Hydraulic head at the junction point: $H_J = 62.98$ m

Flow rates incoming to the junction:

$Q(1) = 0.2883$ m^3/s, $V(1) = 2.294$ m/s
$Q(2) = 0.2887$ m^3/s, $V(2) = 4.084$ m/s
$Q(3) = 0.1388$ m^3/s, $V(3) = 2.828$ m/s
$Q(4) = 0.4288$ m^3/s, $V(4) = 2.696$ m/s
$Q(5) = 0.1728$ m^3/s, $V(5) = 1.375$ m/s

Flow rates outgoing from the junction:

$Q(6) = 0.1719$ m^3/s, $V(6) = 2.432$ m/s
$Q(7) = 0.0578$ m^3/s, $V(7) = 1.177$ m/s
$Q(8) = 0.5783$ m^3/s, $V(8) = 3.636$ m/s
$Q(9) = 0.5094$ m^3/s, $V(9) = 4.054$ m/s

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